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ILLINOIS STATE UNIVERSITY

An Extension of The Black-Scholes Call Option Pricing Model

Shabnam Shahrezaei

Advisor: Dr. Papa Sissokho

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Abstract

This paper extends the Black-Scholes (B-S) call option pricing model and compare the accuracy of the derived B-S model with original model. It expands the Taylor's series and utilizes the Ito's lemma to derive the third moment of Brownian motion. Then it tests the accuracy of the B-S model with including the third moment by predicting the European call option of International Business Machine (IBM) and American Telephone and Telegraph (ATT) for March 2021. For the purpose of comparison, I also test the model for IBM from December 31, 1975 to December 31, 1976. All data is derived from Yahoo finance and Wall Street Journal.

1 Introduction

The Black-Scholes model has been playing a very important role in the financial market. The main purpose of this model is to construct a riskless portfolio taking positions in bonds (cash), option, and the underlying stock (9). An option is a derivative, a contract that gives the buyer the right, but not the obligation, to buy or sell the underlying asset by a certain date (expiration date) at a specified price (strike price). There are two types of options: call options and put options (5).

Calls give the buyer the right, but not the obligation, to buy the underlying asset at the strike price specified in the option contract. Investors buy calls when they believe the price of the underlying asset will increase and sell calls if they believe it will decrease (7). Puts give the buyer the right, but not the obligation, to sell the underlying asset at the strike price specified in the contract (7). The writer (seller) of the put option is obligated to buy the asset if the put buyer exercises their option. Investors buy puts when they believe the price of the underlying asset will decrease and sell puts if they believe it will increase. Also, we have American-style options which can be exercised at any time prior to their expiration and European-style options which can only be exercised on the expiration date. I work with European style call option in this paper because it is easier to work.

In order to illustrate how call option works, I bring a sensible example. Assume a stock option is for 100 shares of the underlying stock. Assume a trader buys one call option contract on ABC stock with a strike price of $\$25$. He pays $\$150$ for the option. On the option's expiration date, ABC stock shares are selling for $\$35$. The buyer/holder of the option exercises his right to purchase 100 shares of ABC at

$\$25$ a share (the option's strike price). He immediately sells the shares at the current market price of $\$35$ per share. He paid $\$2,500$ for the 100 shares ($\$25 * 100$) and sells the shares for $\$3,500$ ($\$35 * 100$). His profit from the option is $\$1,000$ ($\$3,500 - \$2,500$), minus the $\$150$ premium paid for the option. Thus, his net profit, excluding transaction costs, is $\$850$ ($\$1,000 - \$150$). That's a very nice return on investment for just a $\$150$ investment.

So, the question is: what is a fair price to charge for the option? The Black-Scholes formula gives the price of the option, in terms of other quantities, which are assumed known. This mathematical tool is used to calculate the theoretical value of options using current stock prices, expected dividends, the option's strike price, expected interest rates, time to expiration and expected volatility. In this paper, we are going to expand the Black-Scholes, B-S, model and compare the market prices of call options with prices predicted by the expanded B-S option pricing model.

The B-S model for a call option is given by:

$$C_0 = S_0 N(d_1) - X e^{-rT} N(d_2) \tag{1}$$

which has been discussed extensively in the literature. In this equation C_0 is European call option, S_0 is stock price, X is exercise price, r is risk-free interest rate and T is time to expiration. The function $N(\cdot)$ is the cumulative distribution function for a standard normal distribution. The probability that your random variable is less than or equal to x is $0 < N(x) < 1$. The inputs d_1 and d_2 are defined as follows:

$$d_1 = \frac{ln(\frac{S_0}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{ln(\frac{S_0}{X}) + (r - \frac{(\sigma)^2}{2})T}{\sigma\sqrt{T}}$$

where σ is the volatility. For any time interval of length d, the return on the underlying security has a normal distribution with variance $\sigma^2 d$. The first part of the equation 1, $S_0 N(d_1)$, is what we are going to get which is being weighted by some type of a probability. The second part is what we are going to pay. The question is are we going to exercise our option?

It makes sense if the stock price worth more than the exercise price. The value of the call option would be the value of the stock minus the exercise price discounted back today. The higher the S_0, the more likely people would exercise their option. e^{-rT} is discounting or the present value of the exercise price. When the volatility, σ, goes up, d_1 increases and d_2 decreases. As a result, $S_0N(d_1)$ goes up and $Xe^{-rT}N(d_2)$ goes down (i.e., the amount that we get increases and the amount that we pay decreases) so that the value of the call option increases. The B-S model is based on certain assumptions: (i) assuming stocks pay no dividends, (ii) it assumes stock prices follow a random walk, (iii) it assumes no commissions and transactions costs, (iv) the interest rate is constant, and (v) the volatility is constant over time.

Definition 1. *We say that a random process $(X_t : t \geq 0)$, is a Brownian motion with parameters (μ, σ) if:*

1. *For $0 < t_1 < t_2 < \cdots < t_{n-1} < t_n$, the increments $(X_{t_2} - X_{t_1}), (X_{t_3} - X_{t_2}), \cdots, (X_{t_n} - X_{t_{n-1}})$ are mutually independent.*

2. *For $S > 0$, we have $X_{t+S} - X_t \sim N(\mu S, \sigma^2 S)$, meaning increment has a normal distribution with mean μS and variance $\sigma^2 S$.*

3. *X_t is a continuous function of t. We say X_t is a Brownian Motion with drift μ and volatility σ, denoted by $B(\mu, \sigma)$.*

At $\mu = 0$ and $\sigma = 1$, we have a standard Brownian motion, W_t (3), which is zero at initial time ($t = 0$) (4).

Lemma 1. *If $X_t = f(B_t)$ is a function of Brownian Motion, then its differential is as follows:*

$$dX_t = f^{'}(B_t)dB_t + \frac{1}{2}f^{''}(B_t)\boldsymbol{dt} \tag{2}$$

in which we assumed that the function is twice continuously, and differentiable, $f \in C^2$. This stochastic chain rule is known as Ito's Lemma (3).

Proof. The integrated form is as follows,

$$\Delta X_t = X_t - X_0 = f(B_t) - f(B_0) = \sum_{k=1}^{n}[f(B_{t_k}) - f(B_{t_{k-1}})]. \tag{3}$$

where $[0, t]$ is divided into n intervals $[t_0, t_1], ..., [t_{n-1}, t_n]$, $t_0 = 0$ and $t_n = t$.
Using Taylor's series, $f(b) - f(a)$=$f'(a)(b - a)$+$\frac{1}{2}f''(c)(b - a)^2$ for some c with $a < c < b$.
In terms of Brownian Motion, we have

$$f(B_{t_k}) - f(B_{t_{k-1}}) = f'(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}) + \tfrac{1}{2}f''(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})^2.$$

Put it in equation 3:

$$X_t - X_0 = \textstyle\sum_{k=1}^{n}[f'(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}) + \frac{1}{2}f''(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})^2]$$
$$= \textstyle\sum_{k=1}^{n}[f'(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})] + \frac{1}{2}\sum_{k=1}^{n}[f''(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})^2]$$

where n is large and n $\to \infty$. Then,

$$X_t - X_0 = \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)dB_s^2 \tag{4}$$

since $X_t = 0$ at $t = 0$, $f'(B_0)$ and $f''(B_0)$ would be zero. By taking the derivative on both sides in 4 and using the fundamental theorem of calculus, equation 4 is equal to equation 2. Also, $dB_t^2 = dt$ which has proved in appendix (51). □

2 Methodology

2.1 Taylor's Series

In Lemma 1, we started looking at writing down a power series representation of a function of Brownian Motion. So, what we need to do is come up with a more general method for writing a power series representation for a function. We consider two assumptions. The first assumption is that the function $f(x)$ has a power series representation about $x = a$ given by

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + ... \tag{5}$$

where c_n are constants. Second, we need to assume that the function, $f(x)$, has derivatives of every order and that we can in fact find them all (1). Now that we have assumed a power series representation exists, we need to determine what the coefficients, c_n, are. First, consider the case where $x = a$. This gives,

$$f(a) = c_0.$$

All the terms except the first are zero. In order to find any of the other coefficients, we can take the derivative of the function (and its power series) then plug in $x = a$. We get,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + ...$$
$$f'(a) = c_1.$$

Now we can continue to get the second derivative,

$$f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + ...$$
$$f''(a) = 2c_2$$

which leads us to

$$c_2 = \tfrac{f''(a)}{2}.$$

Using the third derivative gives,

$$f'''(a) = 6c_3$$

which leads us to

$$c_3 = \tfrac{f'''(a)}{3!}.$$

In general, a similar calculation yields

$$c_n = \tfrac{f^{(n)}(a)}{n!}.$$

So, if a power series representation for the function $f(x)$ about $x = a$ exists, then the Taylor series (2) for $f(x)$ about $x = a$ is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+f''(a)\frac{(x-a)^2}{2!}+f'''(a)\frac{(x-a)^3}{3!}+... \tag{6}$$

Assume that we have a smooth function f and a Brownian motion, B_t, such that

$$f(B_t) = B_t^2.$$

Consider equation 6 with $x = B_t$ and $a = B_t - dB_t$. Then by taking the derivative of equation 6, we have the Taylor expansion of $f(B_t)$ for some smooth f, as follows

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)(dB_t)^2 + \frac{1}{3!}f'''(B_t)(dB_t)^3 + ... \tag{7}$$

With respect to the second property of Brownian motion 1, we can say dB_t has a standard normal distribution with mean zero and variance dt. Then, the expected value of the second order, $(dB_t)^2$, is approximately equal to the value dt. This value is large enough to be relevant. By the same procedure, we are going to show that the expected value of the third order, $E(|dB_t|^3)$ has the size $(dt)^{\frac{3}{2}}$ which is negligible compared with dt, since dt is much smaller than 1. However, in this study, I want to include in the Taylor's series the third derivative in order to answer the following questions:

Research Questions:

1. ***How to construct a new B-S model by including the third moment?***
2. ***Does this extra term help us to have a more accurate prediction of stock prices in the future?***

Lemma 2. *Suppose we have some function f of Brownian motion, say $f(B_t) = B_t^2$. Then we have,*

$$\int_0^t d(B_s^2) = 2\int_0^t B_s d(B_s) \tag{8}$$

and

$$B_t^2 = 2\int_0^t B_s d(B_s). \tag{9}$$

Proof. We want to create the stochastic differential $df_t = d(B_t^2)$. By using a chain rule, we have,

$$d(B_s^2) = 2B_s d(B_s). \tag{10}$$

Taking the integral from both sides of equation 10, it gives,

$$\int_0^t d(B_s^2) = 2\int_0^t B_s d(B_s). \tag{11}$$

Then $\int df = f$ implies,

$$[B_s^2]_0^t = 2\int_0^t B_s d(B_s). \tag{12}$$

By putting in the bounds, we obtain

$$B_t^2 - B_0^2 = 2\int_0^t B_s d(B_s). \tag{13}$$

Since B_t is a Standard Brownian motion with mean 0, it is equal to zero at the starting point, $t = 0$. So, equation 13 is

$$B_t^2 = 2\int_0^t B_s d(B_s) \tag{14}$$

$\square$

Dividing up the time interval [0,t] into sub-intervals of equal size $\frac{t}{n}$, we can approximate the integral in equation 14 as

$$2\int_0^t B_s d(B_s) \approx 2\sum_{i=0}^{n-1} B\left(\frac{it}{n}\right)\left(B\left(\frac{(i+1)t}{n}\right) - B\left(\frac{it}{n}\right)\right). \tag{15}$$

The difference term is the increments of the Brownian motion from one particular partition point to the next. Based on the Brownian motion properties 1, this increment is independent of the Brownian motion up to that point, or up to $B(\frac{it}{n})$. Also, this increment has mean zero. So, the summation consists of terms with zero mean, forcing it to have zero mean. However, B_t^2 has mean t, because of the variance structure of Brownian motion, so $2B_t dB_t$ cannot be a differential of B_t^2, because its integral does not even have the right expectation, t. However, in order to get the right differential, we can start with $(dB_t)^2$ and model it as

$$\int_0^t (dB_s)^2 \approx \sum_{i=1}^{n} \left(B(\frac{ti}{n}) - B(\frac{t(i-1)}{n}) \right)^2 . \tag{16}$$

Let's define $Z_{n,i}$ as,

$$Z_{n,i} = \frac{B(\frac{ti}{n}) - B(\frac{t(i-1)}{n})}{\sqrt{\frac{t}{n}}} \tag{17}$$

then for each n, the sequence Z_{ni} is a set of identically independently distribution (iid) normal $N(0,1)$. So, we can rewrite equation 16 as

$$\int_0^t (dB_s)^2 \approx t \sum_{i=1}^{n} \frac{Z_{n,i}^2}{n}. \tag{18}$$

Considering the law of large numbers, the right-hand side summation converges to the expected value of Z_{ni}, which is 1. Therefore, we may assume $\int_0^t (dB_s)^2 = t$, or in differential form $(dB_t)^2 = dt$. Then neglecting the orders higher than 2, equation 7 changes to

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2} f''(B_t)dt + 0. \tag{19}$$

Next, we consider B_t^2. We can apply Ito's lemma with $X = B_t$ and $f(B_t) = B_t^2$ and we have

$$d(B_t^2) = 2B_t dB_t + dt \; or \; B_t^2 = 2 \int_0^t B_s d(B_s) + t \tag{20}$$

which at least has the right expectation. In general if X is still a Brownian motion B, then by lemma 1,

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt. \tag{21}$$

Take $(dB_t)^3$, given the partitioning of $[0,t]$ into sub-intervals of equal size $\frac{t}{n}$, we can approximate the integral of $(dB_t)^3$ as

$$\int_0^t (dB_s)^3 = \sum_{i=1}^{n} \left(B(\frac{ti}{n}) - B(\frac{t(i-1)}{n}) \right)^3. \tag{22}$$

But if we let $Z_{n,i}^3$ be

$$Z_{n,i}^3 = \left(\frac{B(\frac{ti}{n}) - B\left(\frac{t(i-1)}{n}\right)}{\sqrt{\frac{t}{n}}} \right)^3 = \frac{\left(B(\frac{ti}{n}) - B(\frac{t(i-1)}{n})\right)^3}{\frac{t}{n}\sqrt{\frac{t}{n}}}. \tag{23}$$

We can rewrite the value for the integral of $(dB_t)^3$ as

$$\int_0^t (dB_s)^3 \approx t\sqrt{t} \sum_{i=1}^{n} \frac{Z_{n,i}^3}{n\sqrt{n}}. \tag{24}$$

As a generalization, we can write it for the kth moment

$$\int_0^t (dB_s)^k \approx t^{\frac{k}{2}} \sum_{i=1}^{n} \frac{Z_{n,i}^k}{n^{\frac{k}{2}}}. \tag{25}$$

But since $(dB_t)^k$ gets much smaller as k increases, we ignore the contribution of $(dB_t)^k$, when $k \leq 3$.

Definition 2. *A (standard, one-dimensional) Brownian motion is a continuous, adapted process $B = [B_t, \mathcal{F}_t; 0 \leq t < \infty]$, defined on some probability space $(\Omega, \mathcal{F}, P)$, with the properties that $B_0 = 0$ a.s. and for $0 \leq s < t$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$ and is normally distributed with mean zero and variance $t - s$ (6).*

Lemma 3. *The expression* $\sum_{i=1}^{n} \frac{Z_{n,i}^{\ 3}}{n\sqrt{n}}$ *can be bounded as follows,*

$$\frac{1}{\sqrt{t}} \leq \sum_{i=1}^{n} \frac{Z_{n,i}^{\ 3}}{n\sqrt{n}} \leq \frac{t\sqrt{t}}{n^2} \tag{26}$$

Proof. Previously, we defined $Z_{n,i} = \frac{B(\frac{ti}{n})-B(\frac{t(i-1)}{n})}{\sqrt{\frac{t}{n}}}$. Since $B(\frac{ti}{n})$ and $B(\frac{ti-t}{n})$ are the standard Brownian motions (Wiener process 3), they have the variances $\frac{ti}{n}$ and $\frac{ti-t}{n}$, respectively. So, $Z_{n,i}$ is,

$$Z_{n,i} \cong \frac{\frac{ti}{n} - \frac{ti-t}{n}}{\sqrt{\frac{t}{n}}} \leq \frac{\frac{t}{n}}{\sqrt{\frac{t}{n}}} = \frac{\sqrt{t}}{\sqrt{n}} \tag{27}$$

then we have,

$$\sum_{i=1}^{n} \frac{Z_{n,i}^{\ 3}}{n\sqrt{n}} \leq n. \left(\frac{\frac{t}{n} . \frac{\sqrt{t}}{\sqrt{n}}}{n\sqrt{n}} \right), \tag{28}$$

which gives,

$$\sum_{i=1}^{n} \frac{Z_{n,i}^{\ 3}}{n\sqrt{n}} \leq n. \left(\frac{t\sqrt{t}}{n^3} \right) = \frac{t\sqrt{t}}{n^2}. \tag{29}$$

Next step is to find the lower bound. Using Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right) \tag{30}$$

with $a_i = \left(\frac{Z_{n,i}}{\sqrt{n}} \right)^{\frac{1}{2}}$ and $b_i = \left(\frac{Z_{n,i}}{\sqrt{n}} \right)^{\frac{3}{2}}$ yields,

$$\left(\sum_{i=1}^{n}\frac{{Z_{n,i}}^{2}}{n}\right)^{2}=\left(\sum_{i=1}^{n}\left(\frac{Z_{n,i}}{\sqrt{n}}\right)^{\frac{1}{2}}\cdot\left(\frac{Z_{n,i}}{\sqrt{n}}\right)^{\frac{3}{2}}\right)^{2} \tag{31}$$

$$\left(\sum_{i=1}^{n}\frac{{Z_{n,i}}^{2}}{n}\right)^{2}\leq\left(\sum_{i=1}^{n}\frac{Z_{n,i}}{\sqrt{n}}\right)\cdot\left(\sum_{i=1}^{n}\frac{{Z_{n,i}}^{3}}{n\sqrt{n}}\right). \tag{32}$$

In the next step, we use $\int_0^t dB_s$ to approximate the value for $\sum_{i=1}^{n}\frac{Z_{n,i}}{\sqrt{n}}$. We have

$$\int_0^t dB_s=\sum_{i=1}^{n}\left(B(\frac{ti}{n})-B\left(\frac{t(i-1)}{n}\right)\right). \tag{33}$$

Since $Z_{n,i}=\frac{B(\frac{ti}{n})-B(\frac{t(i-1)}{n})}{\sqrt{\frac{t}{n}}}$, then equation 33 becomes,

$$\int_0^t dB_s\cong\left(\sum_{i=1}^{n}\frac{Z_{n,i}}{\sqrt{n}}\right)\sqrt{t} \tag{34}$$

then we have,

$$\sum_{i=1}^{n}\frac{Z_{n,i}}{\sqrt{n}}\cong\frac{1}{\sqrt{t}}\int_0^t dB_s \tag{35}$$

then we can rewrite $\int_0^t dB_s$ as,

$$\sum_{i=1}^{n}\frac{Z_{n,i}}{\sqrt{n}}\cong\frac{1}{\sqrt{t}}(B_t-B_0). \tag{36}$$

Based on standard Brownian motion properties 1, B_t at initial time, $t=0$, is equal to zero, and we have,

$$\sum_{i=1}^{n}\frac{Z_{n,i}}{\sqrt{n}}\cong\frac{B_t}{\sqrt{t}}. \tag{37}$$

Based on definition 2, the variance of B_t is $t-s$, which in this situation, $s=0$ and B_t approaches to t as n increases. Then, we have

$$\sum_{i=1}^{n} \frac{Z_{n,i}}{\sqrt{n}} \cong \sqrt{t}. \tag{38}$$

Getting back to equation 32, the left-hand side would goes to 1 as n increases, and with equation 38, we have

$$\frac{1}{\sqrt{t}} \leq \sum_{i=1}^{n} \left(\frac{Z_{n,i}^{\ 3}}{n\sqrt{n}} \right) \tag{39}$$

$\square$

We have proved the lemma. Now if $n \sim t$, e.g., by dividing $[0,t]$ into $n=t$ units, equation 39 would be $\sum_{i=1}^{n} \frac{Z_{n,i}^{\ 3}}{n\sqrt{n}} \cong \frac{1}{\sqrt{t}}$. So, we can take

$$\sum_{i=1}^{n} \frac{Z_{n,i}^{\ 3}}{n\sqrt{n}} \cong \frac{1}{\sqrt{t}} \tag{40}$$

then we have,

$$\int_0^t (dB_s)^3 = t\sqrt{t} \left(\sum_{i=1}^{n} \frac{Z_{n,i}^{\ 3}}{n\sqrt{n}} \right) = t\sqrt{t} \cdot \frac{1}{\sqrt{t}} = t \tag{41}$$

or in differential form $(dB_t)^3 = dt$.
Now we are going to include the third moment of Taylor's series in the Black-Scholes Model. The stock follows an exponential Brownian motion,

$$S_t = exp(\sigma W_t + \mu t). \tag{42}$$

Then we can write Y_t as

$$Y_t = \sigma W_t + \mu t \tag{43}$$

therefore,

$$S_t = e^{Y_t}.$$

Then the stochastic differential equation (SDE) for Y_t is

$$dY_t = \sigma dW_t + \mu dt. \tag{44}$$

By Taylor's series, we have

$$dS_t = \frac{dS}{dy}dy + \frac{1}{2}\frac{d^2S}{dy^2}(dy)^2 + \frac{1}{3!}\frac{d^3S}{dy^3}(dy)^3 \tag{45}$$

expanding equation 45 yields

$$dS_t = \frac{dS}{dy}(\sigma dW_t + \mu dt) + \frac{1}{2}\frac{d^2S}{dy^2}(\sigma^2 dt) + \frac{1}{3!}\frac{d^3S}{dy^3}(\sigma^3 dt). \tag{46}$$

Let's have $(dt)^2 = 0$, $dW_t dt = 0$, $(dW_t)^2 = dt$, $(dW_t)^3 = dt$. Then equation 46 is simplified as,

$$dS_t = \frac{dS}{dy}\sigma dW_t + \left(\mu\frac{dS}{dy} + \frac{1}{2}\frac{d^2S}{dy^2}\sigma^2 + \frac{1}{3!}\frac{d^3S}{dy^3}\sigma^3\right)dt. \tag{47}$$

The exponential function is particularly pleasant as $f'(Y_t) = f''(Y_t) = f'''(Y_t) = f(Y_t) = S_t$. So, we can rewrite the stochastic differential like

$$dS_t = S_t\sigma dW_t + \left(\mu + \frac{\sigma^2}{2} + \frac{\sigma^3}{3!}\right)S_t dt. \tag{48}$$

Equation 48 holds when the interest rate is zero. However, in the real world, we cannot assume that interest rate is zero. When r is not zero, we should expect the growth of cash. In order to remove the growth of cash we need to discount everything. So, by introducing the discount process B_t^{-1}, we can write down the discounted stock as $Z_t = B_t^{-1} S_t$ and a discounted claim $B_t^{-1} X$. Then the new SDE, including non-zero interest rate can be written as:

$$dZ_t = Z_t \sigma dW_t + \left(\mu - r + \frac{\sigma^2}{2} + \frac{\sigma^3}{3!} \right) Z_t dt. \tag{49}$$

2.2 Black-Scholes Formula

Now we are going to write down the Black-Scholes formula for call option with respect to the third moment. As we defined call option before, it is the right but not the obligation to buy a unit of stock for a predetermined amount at a particular exercise date, say T. Let k be the predetermined amount (strike price), then our object would be $max(S_T - k, 0)$ or $(S_T - k)^+$. Based on equation 1, B-S model would be

$$C_0 = S_0 N \left(\frac{log(\frac{S_0}{k}) + \left(r + \frac{\sigma^2}{2} + \frac{\sigma^3}{3!} \right) T}{\sigma \sqrt{T}} \right) - k e^{-rT} N \left(\frac{log(\frac{S_0}{k}) + \left(r - \frac{\sigma^2}{2} - \frac{\sigma^3}{3!} \right) T}{\sigma \sqrt{T}} \right). \tag{50}$$

This is the Black-Scholes formula for pricing European call options with considering the new drift term as $(r + \frac{\sigma^2}{2} + \frac{\sigma^3}{3!})$. In the next section, we are going to test the accuracy of our new B-S model and comparing the result with previous results.

3 Data Analysis and Discussion

James and Larry (8) published a paper using the sample consists of daily closing prices of all call options traded on the Chicago Board of Trade Options Exchange for International Business Machines (IBM) from December 31, 1975 to December 31, 1976. They tried to empirically examine the B-S call option pricing model. For the purpose of comparison, I used the same IBM data set for the same time period. Option prices and prices of the stocks are taken from the Wall Street Journal. For each option expiration date, we have a different riskless rate. The riskless rates are within 1/2 of 1 percent of one another and given the lack of sensitivity of the call price to the riskless rate, our results would be virtually identical had we used a single riskless return for a Treasury Bill with, say, one year to maturity for all expiration date.

For each day, t, a numerical search routine is used to calculate an implied value of σ for each option price. The numerical search routine finds implied values of σ in the interval $.0001$ to $.06$. The range $.0001$ to $.06$ is large enough to include all reasonable values of σ. Table 1 includes these values. I get these data from the James and Larry (8).

Consider the IBM options traded on June 14. Panel A of Table 2 contains the market prices of the options, panel B contains the implied values of σ, panel C contains the B-S model prices based on an estimated value of σ, and panel D includes the B-S model prices with the third moment. The $240 January option has a market price $3.28 greater than the B-S model price while it is only $0.32 greater than our new B-S model price. As we can see from the results in table 2, our new B-S model option pricing is working better than the previous B-S model. The market price is only $\$0.32$ larger than the new B-S model price. The difference between results from previous and new B-S model is significant. By considering the third moment of the Brownian motion, we can conclude that our model becomes more precise.

I also consider the recent data for daily closing prices of all call options for IBM and American Telephone and Telegraph (ATT) for March 2021. This data is derived from Yahoo Finance. Yahoo finance provides the daily implied volatility for each call options. I used forward dividend and yield as a proxy for risk free interest rate. Table 3 and 4 include this data. Panel A in table 3 indicates the market prices of call option for IBM in March 2021. Panel B shows the implied values of IBM. Implied values are different for each day. It is ranged from 0.00 to 0.66. Panel C indicates the predicted call option price calculating by original Black-Scholes model. Panel D includes the call option prices for IBM predicted by the new Black-Scholes model including the third moment. Consider the date $03/23/2021$ in table 3. The market price of IBM for this specific date is $\$16.25$. The predicted call option price from original Black-Scholes model is $\$20.75$ and the predicted call option price from the new Black-Scholes model is $\$16.49$. It is obvious that the Black-Scholes model including the third moment of Brownian motion predicted the market price of call option better than the Black-Scholes model ignoring the higher moments.

Table 4 does the same thing as table 3 for the ATT call option prices. Panel B shows the implied values which is ranged from 0.65 to 2.51. As we can see from the result, both Black-Scholes models have poor predictions for ATT call option prices. However, the new Black-Scholes model including the third moment of

Brownian motion in panel D, did a better job. Consider the date $03/22/2021$. The market price for call options of ATT in panel A is $\$7.90$. The predicted price by original B-S model is $\$11.24$ and by the new B-S model is $\$8.54$. The difference between the B-S model price and the market price is $\$3.34$; however, the difference between the new B-S model price and the market price is $\$0.64$.

Table 1: Sample Implied Values of Sigma for IBM

Date	Exercise Price	January	April	July	October	Stock Price
1006.1976	200.00	0.0	0.0	0.02103	0.0	255.12
1006.1976	220.00	0.0	0.0	0.01730	0.01655	255.12
1006.1976	240.00	0.01412	0.0	0.01362	0.01423	255.12
1006.1976	260.00	0.01263	0.0	0.01118	0.01254	255.12
1006.1976	280.00	0.01154	0.0	0.00980	0.01164	255.12
1106.1976	200.00	0.0	0.0	0.02900	0.0	257.75
1106.1976	220.00	0.0	0.0	0.02087	0.01674	257.75
1106.1976	240.00	0.01431	0.0	0.01567	0.01496	257.75
1106.1976	260.00	0.01304	0.0	0.01080	0.01272	257.75
1106.1976	280.00	0.01158	0.0	0.00979	0.01153	257.75
1406.1976	200.00	0.0	0.0	0.02841	0.0	260.25
1406.1976	220.00	0.0	0.0	0.01931	0.01869	260.25
1406.1976	240.00	0.01540	0.0	0.01472	0.01542	260.25
1406.1976	260.00	0.01289	0.0	0.01066	0.01282	260.25
1406.1976	280.00	0.01169	0.0	0.01016	0.01164	260.25
1506.1976	200.00	0.0	0.0	0.02865	0.0	259.00
1506.1976	220.00	0.0	0.0	0.02413	0.01727	259.00
1506.1976	240.00	0.01498	0.0	0.01588	0.01502	259.00
1506.1976	260.00	0.01316	0.0	0.01072	0.01317	259.00
1506.1976	280.00	0.01173	0.0	0.01063	0.01174	259.00
1606.1976	200.00	0.0	0.0	0.03368	0.0	262.37
1606.1976	220.00	0.0	0.0	0.02542	0.01947	262.37
1606.1976	240.00	0.01519	0.0	0.01630	0.01584	262.37
1606.1976	260.00	0.01287	0.0	0.01068	0.01303	262.37
1606.1976	280.00	0.01172	0.0	0.01018	0.01132	262.37
1706.1976	200.00	0.0	0.0	0.02794	0.0	267.50
1706.1976	220.00	0.0	0.0	0.01398	0.01490	267.50
1706.1976	240.00	0.01358	0.0	0.01108	0.01335	267.50
1706.1976	260.00	0.01248	0.0	0.00820	0.01220	267.50
1706.1976	280.00	0.01140	0.0	0.00983	0.01097	267.50
1806.1976	200.00	0.0	0.0	0.03929	0.0	266.25
1806.1976	220.00	0.0	0.0	0.02627	0.01835	266.25
1806.1976	240.00	0.01463	0.0	0.01790	0.01568	266.25
1806.1976	260.00	0.01323	0.0	0.01087	0.01299	266.25
1806.1976	280.00	0.01206	0.0	0.01153	0.01182	266.25
2106.1976	200.00	0.0	0.0	0.03496	0.0	270.50
2106.1976	220.00	0.0	0.0	0.02514	0.01650	270.50
2106.1976	240.00	0.01374	0.0	0.01733	0.01632	270.50
2106.1976	260.00	0.01305	0.0	0.01254	0.01324	270.50
2106.1976	280.00	0.01207	0.0	0.01186	0.01196	270.50
2206.1976	200.00	0.0	0.0	0.04245	0.0	268.75
2206.1976	220.00	0.0	0.0	0.02312	0.02017	268.75
2206.1976	240.00	0.01658	0.0	0.01412	0.01346	268.75
2206.1976	260.00	0.01336	0.0	0.01066	0.01228	268.75
2206.1976	280.00	0.01153	0.0	0.01133	0.01137	268.75
2306.1976	200.00	0.0	0.0	0.03953	0.0	271.50
2306.1976	220.00	0.0	0.0	0.02708	0.01796	271.50
2306.1976	240.00	0.01309	0.0	0.01883	0.01555	271.50
2306.1976	260.00	0.01317	0.0	0.01119	0.01374	271.50
2306.1976	280.00	0.01212	0.0	0.01199	0.01203	271.50

Table 2: IBM Option Data for June 14, 1976
(S* is the current stock price)

Panel	Exercise Price	January	July	October
		Market Prices		
Panel A	$200		$62	
	$220		$42	$47.50
	$240	$36.50	$23	$30.50
	$260	$22	$7.13	$16.13
	$280	$12	$0.94	$7
		Implied Values of Sigma		
Panel B	$200		0.028412	
	$220		0.019312	0.018691
	$240	0.015404	0.014725	0.015417
	$260	0.012885	0.010658	0.012824
	$280	0.011689	0.010158	0.01164
		Black-Scholes Model Prices		
Panel C	$200		$61.23	
	$220		$48.38	$43.90
	$240	$33.22	$22.53	$28.13
	$260	$21.95	$8.42	$16.15
	$280	$13.73	$1.90	$8.28
		Black-Scholes Model Prices With Third Moment		
Panel D	$200		$62.36	
	$220		$42.57	$44.10
	$240	$36.18	$22.78	$28.54
	$260	$22.05	$7.05	$16.28
	$280	$14.25	$1.36	$7.75
	r(Annual)	5.90%	5.30%	5.60%
	S*	$255.85	$260.25	$258.03

Table 3: IBM Option Data for March, 2021
(S* is the current stock price)

Panel	Exercise Price	March	Date
		Market Prices	
Panel A	$114	16.93	3/24/2021
	$115	16.25	3/23/2021
	$145	0.02	3/22/2021
	$142	7.95	3/19/2021
	$110	19.40	3/18/2021
	$118	10.80	3/17/2021
	$150	0.03	3/15/2021
		Implied Values of Sigma	
Panel B	$114	0.00	3/24/2021
	$115	0.66	3/23/2021
	$145	0.46	3/22/2021
	$142	0.38	3/19/2021
	$110	0.26	3/18/2021
	$118	0.35	3/17/2021
	$150	0.31	3/15/2021
		Black-Scholes Model Prices	
Panel C	$114	17.83	3/24/2021
	$115	20.75	3/23/2021
	$145	0.03	3/22/2021
	$142	2.07	3/19/2021
	$110	21.86	3/18/2021
	$118	14.98	3/17/2021
	$150	0.61	3/15/2021
		Black-Scholes Model Prices with Third Moment	
Panel D	$114	17.84	3/24/2021
	$115	16.49	3/23/2021
	$145	0.07	3/22/2021
	$142	8.87	3/19/2021
	$110	21.90	3/18/2021
	$118	13.33	3/17/2021
	$150	0.03	3/15/2021
	r(Annual)	0.05	
	S*	131.27	3/24/2021

Table 4: ATT Option Data for March, 2021
(S* is the current stock price)

Panel	Exercise Price	March	Date
		Market Prices	
Panel A	$20	4.25	3/24/2021
	$22	7.90	3/22/2021
	$20	10.00	3/19/2021
	$35	0.01	3/18/2021
	$25	4.80	3/15/2021
	$26	5.16	3/8/2021
		Implied Values of Sigma	
Panel B	$20	0.89	3/24/2021
	$22	2.00	3/22/2021
	$20	2.51	3/19/2021
	$35	0.65	3/18/2021
	$25	1.09	3/15/2021
	$26	0.99	3/8/2021
		Black-Scholes Model Prices	
Panel C	$20	10.32	3/24/2021
	$22	11.24	3/22/2021
	$20	13.68	3/19/2021
	$35	0.91	3/18/2021
	$25	6.08	3/15/2021
	$26	4.02	3/8/2021
		Black-Scholes Model Prices with Third Moment	
Panel D	$20	10.61	3/24/2021
	$22	8.54	3/22/2021
	$20	10.69	3/19/2021
	$35	0.02	3/18/2021
	$25	5.17	3/15/2021
	$26	4.22	3/8/2021
	r(Annual)	0.07	
	S*	29.99	3/24/2021

4 Conclusion and Recommendations

The main purpose of this research is to mathematically expand the B-S model and empirically analyze it. In the original B-S model, the moments of Brownian motion higher than 2, is considered zero (2). This directly affects the value of inputs d_1 and d_2 in the model. In this research, we have shown that the third moment of Brownian motion can affect the accuracy of the B-S model. We answered our research question in section 2.1. In order to add the third moment, $(dB_t)^3$, we have used Brownian motion properties and Taylor's series. We have shown that $(dB_t)^3$ can be approximately written as $t\sqrt{t}\sum_{i=1}^{n}\frac{Z_{n,i}^{3}}{n\sqrt{n}}$.
Lemma 3 illustrates the boundary of $\sum_{i=1}^{n}\frac{Z_{n,i}^{3}}{n\sqrt{n}}$. Then we prove that $\sum_{i=1}^{n}\frac{Z_{n,i}^{3}}{n\sqrt{n}}$ can be approximately equal to $\frac{1}{\sqrt{t}}$ which makes $(dB_t)^3$ approximately equal to dt. After plugging the third moment into B-S model, I empirically tested it using the daily closing option price of IBM and ATT. As we can see the results in Table 2, 3, and 4, the option prices of the new B-S model in panel D predicts the real market option prices much better than the original B-S model prices in panel C. It indicates the important role of the third moment term in the accuracy of the model. Therefore, it is necessary to include the third moment of the Brownian motion in our model. However, since I have tested the new derived B-S model on only two stock markets over a one-month time period, results drawn from this research must be provisional.
Finally, our analysis sheds some light on the apparently profitable option trading strategy of James and Larry (8). Our B-S model price exceeds their B-S model price, and the differences are too large. For the future studies, I recommend considering the higher moments of these Brownian motion into the model instead of considering them as zero. We derived the general form of $(dB_s)^k$ in equation 25. It will help us to approximate the value for $\sum_{i=1}^{n}\frac{Z_{n,i}^{k}}{n^{\frac{k}{2}}}$. Then, we can derive the new B-S model with considering the higher moments of Brownian motion. Also, I recommend utilizing a broader time period for the stock market data sets. It helps the results to be more reliable.

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6 Appendix

6.1 Ito's Lemma

From Lemma 1, we have

$$X_t - X_0 = \sum_{k=1}^{n}[f^{'}(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})] + \frac{1}{2}\sum_{k=1}^{n}[f^{''}(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})^2]. \quad (51)$$

We want to show that the second term in equation 51 converges to $\frac{1}{2}\sum_{k=1}^{n}[f^{''}(B_{t_{k-1}})(t_k - t_{k-1})]$ as n approaches to infinity by using mean squared technique.

Proof. Let's define X_n as,

$$X_n = \sum_{k=1}^{n}[f^{''}(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})^2]$$

and X as,

$$X = \sum_{k=1}^{n}[f^{''}(B_{t_{k-1}})(t_k - t_{k-1})].$$

Recall that the mean squared convergence means that the expected value of squared of the difference between X_n and X tends to zero as n approaches to infinity. So, we have,

$$E\left[(X_n - X)^2\right] =$$
$$E\left[\left(\sum_{k=1}^{n}[f^{''}(B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}})^2] - \sum_{k=1}^{n}[f^{''}(B_{t_{k-1}})(t_k - t_{k-1})]\right)^2\right]$$
$$= E\left[\left(\sum_{k=1}^{n} f^{''}(B_{t_{k-1}})\left((B_{t_k} - B_{t_{k-1}})^2 - (t_k - t_{k-1})\right)\right)^2\right]$$

This expression looks quit complicated, however, we can utilize a simple trick to make it simpler. Let's recall,

$$E\left[\left(\sum_{k=1}^{n} x_k\right)^2\right] = E\left[\sum_{k=1}^{n} E\left[x_{t_k}^2|F_{t_{k-1}}\right]\right] + 2E\left[\sum_{k=1}^{n}\sum_{j=1}^{k-1} E\left[x_{t_k}x_{t_j}|F_{t_{k-1}}\right]\right]. \quad (52)$$

Now, consider

$$x_{t_k} = f^{''}(B_{t_{k-1}})\left((B_{t_k} - B_{t_{k-1}})^2 - (t_k - t_{k-1})\right)$$

Then put the value of x_{t_k} into equation 52, the second term would be zero and we have,

$$E\left[\left(\sum_{k=1}^{n} x_k\right)^2\right] = E\left[\sum_{k=1}^{n} E\left[x_{t_k}^2 | F_{t_{k-1}}\right]\right] = E\left[\sum_{k=1}^{n} f''^2\left(B_{t_{k-1}}\right) V\left[(B_{t_k} - B_{t_{k-1}})^2 | F_{t_{k-1}}\right]\right] + 0 \tag{53}$$

Since $E\left[(X_n - X)^2\right] = E\left[\left(\sum_{k=1}^{n} x_k\right)^2\right]$, we have

$$E\left[(X_n - X)^2\right] = E\left[\sum_{k=1}^{n} f''^2\left(B_{t_{k-1}}\right) V\left[(B_{t_k} - B_{t_{k-1}})^2 | F_{t_{k-1}}\right]\right] \tag{54}$$

and we know that the variance of the square of the Brownian increments is two times the square of the length of the interval. As the length of each sub-interval is $\frac{t}{n}$, we can write equation 54 as follows,

$$E\left[(X_n - X)^2\right] = E\left[\sum_{k=1}^{n} f''^2\left(B_{t_{k-1}}\right) 2(t_k - t_{k-1})^2\right] = 2\left(\frac{t}{n}\right)^2 E\left[\sum_{k=1}^{n} f''^2\left(B_{t_{k-1}}\right)\right]. \tag{55}$$

Now, if we let n tends to infinity, we have

$$\lim_{n\to\infty} E\left[|X_n - X|^2\right] = 0 \tag{56}$$

Therefore, we have proved that $X_n \to X$ as $n \to \infty$. □

6.2 Wiener Process

We define stochastic process, X_t, as a variable whose value changes over time t in an uncertain way (Discrete/Continuous Time or Discrete/Continuous Variable)

Definition 3. *The Wiener process, W_t, is the stochastic process which fall in the continuous variable and continuous time category. At initial time, $t = 0$, the value for W_0 would be zero. For $t = T$, we have,*

$$W_T = (\epsilon_0 + \epsilon_{\Delta t} + ... + \epsilon_{T-\Delta t}).\sqrt{\Delta t}\ E(W_T) = 0$$
$$V(W_T) = n.\Delta t = T$$
$$W_T \sim N(0, T)$$

Also, the Wiener process has non-overlapping intervals like Brownian motion. So, for $t_1 < t_2 < t_3$, we have $(W_{t3} - W_{t2})$ and $(W_{t2} - W_{t1})$ which are independent. We use these properties of Wiener process where we want to talk about the standard Brownian motion.

6.3 R-Code

In order to get the predicted value of option prices by our new B-S model, I used R. I import the value S which is current stock price, K which is strike price, t is time to expiration, $Sigma$ which is implied volatility, and $rFree$ which is risk free interest rate. Then I define d_1 and d_2 with including the third moment. Finally, I define our new B-S model by C_0 which shows the European call option. C_0 gives us the predicted option prices. It includes the function $N(d_i)$ which is shown by $pnorm$. As we explained previously, $N(d_i)$ is cumulative Normal distribution function.

For table 2, we are going to use the following R code by considering different values of sigma in panel B:

$S < -255$

$K < -200; 220; 240; 260; 280$

$t < -1$

$Sigma < -0.015$

$rFree < -0.059$

$d1 < -(log(S/K) + (rFree + (Sigma^2)/2 + (Sigma^3)/6) * t)/(Sigma * sqrt(t))$

$d2 < -(log(S/K) + (rFree - (Sigma^2)/2 - (Sigma^3)/6) * t)/(Sigma * sqrt(t))$

$C_0 = S * pnorm(d1) - K * exp(-rFree * t) * pnorm(d2)$

For table 3, the following R code is used by considering different values of sigma in panel B:

$S < -131.27$

$K < -114; 115; 145; 142; 110; 118; 150$

$t < -1$

$Sigma < -0.00$

$rFree < -0.06$

$d1 < -(log(S/K) + (rFree + (Sigma^2)/2 + (Sigma^3)/6) * t)/(Sigma * sqrt(t))$

$d2 < -(log(S/K) + (rFree - (Sigma^2)/2 - (Sigma^3)/6) * t)/(Sigma * sqrt(t))$

$C_0 = S * pnorm(d1) - K * exp(-rFree * t) * pnorm(d2)$

Finally,for table 4, the following R code is used by considering different values of sigma in panel B:

$S < -29.99$

$K < -20; 22; 20; 35; 25; 26$

$t < -1$

$Sigma < -0.89$

$rFree < -0.07$

$d1 < -(log(S/K) + (rFree + (Sigma^2)/2 + (Sigma^3)/6) * t)/(Sigma * sqrt(t))$

$d2 < -(log(S/K) + (rFree - (Sigma^2)/2 - (Sigma^3)/6) * t)/(Sigma * sqrt(t))$

$C_0 = S * pnorm(d1) - K * exp(-rFree * t) * pnorm(d2)$